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PROGRAMED MATERIALS IN MATHEMATICS FOR SUPERIOR STUDENTS IN RURAL SCHOOLS.

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A STUDY WAS CONDUCTED ON THE ACHIEVEMENTS OF SUPERIOR STUDENTS USING PROGRAMED INSTRUCTION AND SEMINARS TO STUDY MODERN MATHEMATICS. TWO EXPERIMENTS WERE CONDUCTED TO TEST THE HYPOTHESES THAT (1) SUPERIOR STUDENTS USING PROGRAMED INSTRUCTION WILL ACHIEVE HIGHER SCORES THAN THOSE USING CONVENTIONAL MATERIALS, (2) STUDENTS USING PROGRAMED INSTRUCTION BY MACHINE WILL NOT DIFFER FROM THOSE USING A PROGRAMED TEXTBOOK, AND (3) STUDENTS WILL NOT SHOW A PREFERENCE BETWEEN THE TWO MODES OF PROGRAMED INSTRUCTION. FOUR GROUPS WERE ESTABLISHED FOR THE DASIC EXPERIMENT. THE SAMPLE CONSISTED OF 100 HIGH SCHOOL STUDENTS WHO WERE RANDOMLY ASSIGNED TO THE MACHINE, PROGRAMED TEXTBOOK, CONVENTIONAL TEXTBOOK, OR CONTROL GROUPS. THE RESULTS INDICATED THAT SELF-INSTRUCTIONAL MATERIALS IN MATHEMATICS CAN BE USED PROFITABLY BY SUPERIOR STUDENTS WITH OR WITHOUT A TEACHER. (RS)

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TABLE OF CONTENTS

		Page
I	General Introduction	1
II	Hypotheses	5
III	Outline of the Project	
	Experimental design-Experiment I	6
	The sample	8
	The schedule of testing	9
	Testing points	10
	Experiment II	11
	Types of data gathered	11
	Preparation of the programs	12
IV	Results	
	Experiment I	
	Sets, Relations and Functions	15
	Groups and Fields	16
	Experiment II	
	An Introduction to Probability	18
	The preference scale	20
V	Discussion	
	The primary hypothesis	24
	The secondary hypotheses	27
	The preference scale	31
	Programming technology	32
	A general conclusion	39
References		43
Appendix A - The Teaching Machine		45
Appendix B - The Programs		
	Table of contents for:	
	Sets, Relations, and Functions	48
	Groups and Fields	49
	An Introduction to Probability	51
	Sample frames from:	
	Sets, Relations, and Functions	52
	Groups and Fields	56
	An Introduction to Probability	62
Appendix C - Extension of the Study		68
Appendix D - Criterion Measures		
	Sets, Relations and Functions	72
	Groups and Fields	74
	An Introduction to Probability	79

I. GENERAL INTRODUCTION

The small high schools of the United States have been unable and/or unwilling to provide their talented students with a curriculum designed to challenge and develop these talents; yet, the small high schools, in fact, have in their student bodies a significant proportion of the potentially superior students in the country. The inability of the economy of the rural area to provide the high per-pupil cost associated with a program designed to educate the few rather than the many is one oft-mentioned factor underlying the failure of the small school to provide for these students. Frequently, the quality of the instruction is too low to provide the challenge, for salaries tend to be substandard for teachers - and for school administrators, also - consequently, it is difficult to recruit well-educated and well-trained teachers and administrators capable of inspiring the superior student to raise his level of aspiration and his achievement to that which is characteristic of the gifted student in a typical middle-class, suburban high school. Finally, the persistence of certain values and views held toward education by taxpayers, school boards, school administrators, and many teachers in rural areas prevent the rapid changes in the schools which sometimes are possible in the less well-structured, newer suburban areas. In a sense, the community goals and aspirations severely restrict those students who are capable of excelling the goals held by those who are the pacesetters and rule-makers in their communities.

The inability of the rural school to educate the superior student is apparent if one compares the achievement scores of rural students with those obtained by students from a metropolitan high school on

scholastic aptitude tests. The loss of these talents to our society can only be viewed as alarming.

To be sure, there are notable exceptions to the communities, the teachers and the school administrators described. Under the impetus of an able teacher or administrator, a foundation grant, or a group of lay leaders, some rural secondary schools have developed challenging educational programs for all students. In recent years, there have been widely circulated reports on the efforts to induce changes in some secondary schools. For examples, see:

1. The Beecher Project, Beecher High School, Beecher, Illinois
2. Project on the Academically Talented Student, National Education Association, The Conference Report, February, 1958
3. N.C.A. Superior and Talented Student Project (Carnegie Corporation)
4. Rocky Mountain Project and the Catskill Project (Fund for the Advancement of Education)
5. The Northern New England Educational Resources Project (Goddard College)

Since 1958 Bucknell University, through the Upper Susquehanna Valley Program of Cooperative Research (USVP)[1], has been a partner with more than a dozen rural school systems concerned with the improvement of instruction and the upgrading of curricula. Of the

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[1]The Upper Susquehanna Valley Program was initiated under a grant from the Fund for the Advancement of Education of the Ford Foundation. Until the termination of the grant in 1962, it was supported jointly by the Fund, Bucknell University, and the participating public schools.

many techniques of teaching which could be adapted to the small high school, the two which seem to hold considerable promise for use with the superior student are (a) the seminar conducted by an able college teacher (in rare instances, by a secondary school teacher) and (b) programmed self-instructional material. On a modest level, USVP has conducted since 1959 a weekly seminar on campus in the "Great Books" tradition for twenty-five superior high school seniors from several rural schools. A similar program in mathematics was begun in 1960. For both courses, the instructors have reported that under these conditions talented high school students can be led to a significantly higher level of achievement than they have been exhibiting in their regular public school classes in literature and mathematics.

When the investigation with which this report is concerned was planned in 1959 it seemed probable that programmed instruction promised to be of considerable value in the education of the superior student, particularly as a supplement to classroom instruction. One limitation of the weekly seminar method which we have used in the program for superior students is the difficulty of providing the student with systematic, organized off-campus instructional material which would support the seminar method and which would be inherently motivating for the student who must carry a full program of courses in the public school and find time and energy for preparing for a very rigorous extra course. It seemed possible that programmed materials for use in a teaching machine or in a book format might overcome this limitation. As a consequence, this investigation was intended to explore the use of a combination of the weekly seminar

led by an able teacher and programmed supplements or units in one core subject, mathematics.

Mathematics rather than English literature was selected chiefly because of available talent, investigator interests, and the apparent convenience and ease with which it was believed that mathematics could be programmed. Further impetus for selecting this subject was found in the National Education Association's 1958 report entitled, Mathematics for the Academically Talented Student in the Secondary School, in which it is stated, "In instances where only relatively small number of talented students are identified, all efforts must be made to provide for these students. . . .The possibilities of periodically bringing together even individual students from a broader area than the community . . . to a central location . . . should be investigated." (italics added)

If the approach to instruction described in this report improves the achievement of the participants in mathematics significantly and, indirectly, in related courses, e.g. physics, at a reasonable per-pupil cost, one could expect that

1. some rural school districts would be able and willing to support similar programs for their superior students.
2. it might be possible to generalize from the results of this study to other core subjects, e.g. foreign languages, literature, history, and the sciences.
3. teachers and school administrators would place more confidence in the instructional value of programmed instruction and the weekly seminar.
4. participating students and their parents might change their

educational values and aspirations and, in turn, influence positively the aspirations of non-participants.

II. HYPOTHESES

The hypothesis of major interest which was tested in this investigation is:

> Superior students who use programmed instruction in either machine or text format will achieve significantly higher test scores for the seminar course in topics in modern algebra than will superior students who use conventional textbook material for self-study of the same subject matter.

At the time that this study was conceptualized it was considered desirable to test a hypothesis of secondary interest, viz.

> Students who receive programmed instruction in mathematics through the medium of a machine will not differ significantly in achievement in the seminar course in mathematics from students who receive their instruction through the medium of a programmed textbook.

Several tests of this hypothesis have been reported since 1959 and the results of these studies will be compared with our findings in the section, Discussion.

The primary purpose of this study then, is not a comparison of "traditional" instruction versus programmed instruction, nor is it a comparison of instruction via machine versus instruction via programmed textbook. Instead, the major purpose is to investigate the

contribution of programmed instruction combined with a weekly seminar to achievement in modern mathematics with superior students.

Two other subsidiary hypotheses were investigated during the course of this study. In a second experiment it is hypothesized that,

1. The achievement of Ss who read the confirming response before recording their answers to a teaching frame will not differ significantly from the achievement of Ss who record their answers to a teaching frame before reading the confirming response.
2. Students who have experience with both teaching machine mode and programmed text mode will show no preference for one mode rather than the other.

The first of these hypotheses is derived from research in which typically no significant differences in achievement were found between response modes (Stolurow and Walker, 1962; Moore and Smith, 1962; Feldhusen and Birt, 1962; Evans, Glaser and Homme, 1959; Goldbeck, 1960; Campbell, 1961) although response modes may differ in relative efficiency. The second hypothesis is derived from the observation made frequently by students that machines do not contribute to their learning.

III. OUTLINE OF THE PROJECT

Experimental design - Experiment I

The basic experiment included a sample of one hundred high-school seniors who were randomly assigned to one of four groups:

I. Machine Group

Students worked on programmed materials in mathematics presented through a machine which is described in Appendix A.

II. Programmed Textbook Group

Students worked on programmed materials in mathematics presented in loose-leaf text-format.

III. Conventional Textbook Group

Students worked on mathematics presented in the narrative form usually found in conventional texts.

IV. Control Group

Students were enrolled in high school mathematics course but they did not participate in the seminar nor did they have direct access to any of the instructional materials used in the study.

The three experimental groups met as groups for one hour each day after school in classrooms in the high school which they regularly attended. Each group was monitored by a proctor who provided no information or answered any questions on the content of the learning program or on content of the text material used by one of the experimental groups. The major responsibilities of the proctor were the distribution of materials to his group and the supervision of the group. For example, for the machine group, the proctor distributed the machines each period, provided the learning program for each

machine, assisted in the adjustment of any machine when a malfunction occurred, and collected the answer tapes at the end of each period. No discussions among students of the programmed mathematics or of the textbook material were permitted during class sessions.

Four periods of each week were devoted to self-study and a fifth period was used for lecture and discussion. All seminars were conducted by the head of the department of mathematics in the high school in which the study was conducted to maximize comparability of teaching. Each of the three experimental groups participated in the seminar on a different day of the week in order to keep class size small enough to hold discussions. For reasons which will be presented in the section on Discussion, the seminar for each group was conducted only for the first learning program, Sets, Relations and Functions. For the second learning program, Groups and Fields, no seminar was held; for the third learning program, An Introduction to Probability, the three experimental groups were combined for discussion for one period each week.

The conventional textbook group was given a set of mimeographed materials which in content paralled the content of the programs as closely as was possible, i.e. within the limits of existing text material of a satisfactory level for the Ss in this study, the Ss of this group simply studied these materials an hour each day, while the other groups worked on learning programs.

The Sample

One hundred high school junior and senior students (52 females, 48 males) volunteered to participate in the program. The Ss, each of whom had had a minimum of two years of high school mathematics,

were randomly assigned to one of the four groups. Students were told that they would be given one-half credit toward their graduation requirements following completion of the course in order to provide comparable, extrinsic motivation for all groups. Any student who dropped out of the study after the first six weeks received a failing grade for the course. An intelligence test score and an Iowa Mathematics Achievement score were obtained for each S. These measures were used as control variables in analyses of covariance. Materials were assigned to Ss on a weekly basis and Ss were permitted to work at their own rate. Upon completion of a week's assignment, Ss were allowed to work on other assignments.

The Schedule of Testing

Experiment I

Unit I Sets, Relations, and Functions

Pre-test	September 15, 1961
Test I	September 29, 1961
Test II	October 9, 1961
Test III	October 23, 1961
Test IV	October 30, 1961
Post-test	November 1, 1961
Retention test	March 29, 1962

Unit II Groups and Fields

Pre-test	November 2, 1961
Test I	November 29, 1961
Test II	December 16, 1961
Test III	December 20, 1961

Test IV	February 1, 1962
Test V	February 9, 1962
Post-test	February 12, 1962
Retention test	March 30, 1962

Experiment II

Unit III An Introduction to Probability

Pre-test	February 12, 1962
Test I	February 26, 1962
Test II	March 9, 1962
Test III	March 23, 1962
Post-test	March 28, 1962
Retention test	June 4, 1962

Testing Points

Unit I Sets, Relations and Functions

Test I	after 296 frames
Test II	after 539 frames
Test III	after 724 frames
Test IV	after 1036 frames
Post-test	after 1036 frames

Unit II Groups and Fields

Test I	after 507 frames
Test II	after 723 frames
Test III	after 851 frames
Test IV	after 1078 frames
Test V	after 1426 frames
Post-test	after 1426 frames

Unit III An Introduction to Probability

Test I	after 241 frames
Test II	after 607 frames
Test III	after 907 frames
Post-test	after 970 frames

Experiment II

A minor variation was introduced in the study after the Ss had completed the programmed unit, An Introduction to Probability. The changes were as follows:

(a) Machine group shifted to programmed-text.

(b) Programmed-text group shifted to machine.

(c) Textbook group shifted to programmed-text and instructed to read the confirming response before responding to the frame.

(d) The three groups were combined for one period of discussion each week.

By shifting the Ss who had worked on programs in a machine to a programmed text and vice versa, Ss preference for text versus machine could be determined. By instructing one programmed text group to read the confirming response before responding to a frame while the second programmed text group purportedly responded to the frame before reading the confirming response, a check on the effects of "cheating" on achievement could be obtained. Finally, by combining all groups for the seminar, it was possible to provide better control of the teacher variable.

Types of Data Gathered

As was indicated above, a general achievement test score in

mathematics and a group intelligence test score were obtained for each S. Pre-test and post-test scores were obtained for each S for each of the three units of instruction as were scores on several tests for segments of each of the three units. Retention test scores were obtained for each S as indicated in the Schedule above. The post-tests which were used are presented in Appendix D.

Error rates for each of the units were computed for each experimental group in Experiment I and for two of the groups in Experiment II. Since the third group was instructed to read the confirming response before responding to the frame, error rate was non-existent for this group.

Time per block of frames completed each period was recorded for all Ss who used learning programs.

In the original plan for this study, it was proposed that grades in college courses in mathematics and physics, and the score on SAT Mathematics be included as dependent variables; however, the study was reduced from four years to two years, making it impossible to conduct the necessary follow-up to obtain these data.

All Ss in the experimental groups were administered a preference test for programmed material at the end of the unit on Groups and Fields and at the end of the unit on An Introduction to Probability.

Preparation of the Programs

Beginning in August, 1960 two mathematicians were selected for training as programmers. One was an outstanding teacher of high school mathematics; the other, a highly esteemed teacher of college freshman mathematics. Both teachers had had broad training in modern mathematics. The teachers and the principal investigators planned

three units of instruction for programming. The selection of topics was based on the recommendations of various mathematics study groups, including the School Mathematics Study Group, the Illinois project, and the National Education Association report on mathematics for the academically talented student. After thorough study of the sparse literature on programming, visits were made to the Center for Programed Instruction and to Hollins College to obtain further, direct information on the programming of mathematics. Objectives for each of the three units then were prepared in the form of a detailed tables of content.[1]

As frames were prepared, three high school students worked through them to provide the programmers with information for revision. The program on Sets, Relations, and Functions was given further testing in the Spring, 1961 on a small group of superior high school seniors. The program was revised and tested on a group of high school teachers of mathematics enrolled in an NDEA summer institute. Because of the great difficulty experienced in programming the complex subject matter of mathematical groups and fields, it was not possible to subject the unit on this topic to the same extensive pre-testing. The unit on probability was tested on only five students before it

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[1]These tables are presented in Appendix B in the form in which they evolved for the final revision for each unit. Sample frames for each program are presented, also.

was used in Experiment II.[1]

The programs which were prepared totaled approximately 3300 frames. All programs were linear, constructed-response in type with no branching or looping incorporated. The approach to programming each of these units was largely intuitive rather than a systematic approach such as ruleg or mathetics. The approach was based on the untested assumption that successful teachers intuitively know how to sequence and order material for effective communication with students and that any attempt to impose a mechanical system on their teaching might reduce their effectiveness markedly. As experience was gained in developing, constructing, and testing programmed material, a philosophy of programming which permits the teacher to exercise his ingenuity and utilize his successful experiences, yet which provides some checks and balances on the programmer, has been formulated and tried with some success. This technique, called LTOR, will be described in the section on Programming Technology.

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[1]Non-federal funds were obtained for further development and testing of each of the units. Some of the findings which resulted from the use of the materials in contexts other than those provided in the body of this Report are presented in Appendix C.

IV. RESULTS

Experiment I

In this experiment two units of programmed material were used. The data obtained are presented in tabular form by unit.

Sets, Relations and Functions

(a) Mean time to complete the unit:

machine group	21.9 hours (S.D. 3.6 hrs.)
programmed text group	19.3 hours (S.D. 4.2 hrs.)

(b) Mean error rate on the unit:

machine group	19.1%
programmed text group	13.4%

Note: $t = 1.843$, $df = 29$, $p > .05$

(t must equal 2.045 for significance)

(c) Analyses of variance on test scores for the three experimental groups:

	F	p
Test I	.530	> .05
Test II	1.284	> .05
Test III	1.271	> .05
Test IV	2.270	> .05

An analysis of covariance on the post-test scores, with the scores on the intelligence test and the Iowa Mathematics Achievement Test serving as control variables, resulted in an F of 115.531 ($p < .001$) for all four groups and an F of 2.827 ($p > .05$) for the three experimental groups indicating (1) that Ss in the experimental groups achieved significantly higher scores on the post-test than did the controls but, (2) that the differences in achievement among

the experimental groups on the post-test of the unit were not statistically significant. The machine vs programmed text vs conventional text-self-study variable did not produce differences in achievement, then, on any part-test, on the post-test, nor, as will be indicated, on the retention test.

An analysis of covariance on the scores from the retention test, for the three experimental groups, with the scores from the intelligence test and the Iowa Mathematics Achievement Test serving as control variables, resulted in an F of .548 ($p > .05$), indicating that the amount retained was not differentially affected by the mode of instruction.

Finally, rank-order correlations were computed between the scores on the post-test and the number of errors made on the program. For the machine group the relationship between scores and errors was not significant ($r_s = .347$, $df = 15$, $p > .05$); while, for the programmed text group, errors and post-test scores were related ($r_s = .542$, $df = 17$, $p < .05$) indicating a tendency for high achievement to be related to infrequent commission of errors.

Groups and Fields

(a) Mean time to complete the unit:

machine group	27.2 hours (S.D. 6.1 hrs.)
programmed text group	24.4 hours (S.D. 5.8 hrs.)

(b) Mean error rate on the unit:

machine group	16.2%
programmed text group	14.4%

Note: $t = 1.040$, $df = 33$, $p > .05$

(t must equal 2.040 for significance)

(c) Analyses of variance on test scores for the three experimental groups:

	F	p	Required
Test I	10.100	$<.01$	3.20

t-test between:

a) machine and programmed text .861, $p > .05$

b) machine and conventional text 4.302, $p < .01$

c) programmed text and conventional text 3.430, $p < .01$

Test II	2.614	$>.05$	3.20
Test III	.158	$>.05$	3.20
Test IV	3.258	$<.05$	3.20

t-test between:

a) machine and programmed text .724, $p > .05$

b) machine and conventional text 2.491, $p < .02$

c) programmed text and conventional text, 1.876, $p > .05$

Test V	2.374	$>.05$	3.20

Significant differences among the experimental groups were found for two of the five part-tests. On Test I, the achievement of both the machine and programmed text groups was significantly higher than that of the group which used the conventional text. On Test IV, the machine group achieved significantly higher scores than the conventional textbook group.

An analysis of covariance on the post-test scores, with the scores on the intelligence test and the Iowa Mathematics Achievement Test serving as control variables, resulted in an F of 64.423 ($p < .01$) for all four groups and an F of 12.629 ($p < .01$) for the three experimental groups indicating (1) that the Ss in the experimental groups

achieved significantly higher scores on the post-test than did the controls and, (2) that the experimental groups differed in achievement among themselves. A t-test analysis resulted in one significant comparison; viz., the machine group again was higher than the conventional textbook group ($t = 4.401$, $p < .01$).

An analysis of covariance on the scores from the retention test for this unit, with the scores from the intelligence test and the Iowa Mathematics Achievement Test serving as control variables, resulted in an F of 6.431, $p < .01$. Again, the t-test analysis showed a significant difference in retention between the machine group and the conventional textbook group ($t = 2.973$, $p < .01$).

It should be noted that the control variables are lost by changing from an analysis of covariance. If adjusted means are compared, the programmed text group also showed significantly better retention than the conventional textbook group.

Finally, rank-order correlations were computed between the scores on the post-test and the number of errors made on the program. For the machine group, the relationship between the number of errors committed and post-test scores was significant ($r_s = .484$, $df = 14$, $p < .05$); while, for the programmed text group the relationship was non-significant ($r_s = .178$, $df = 13$, $p > .05$).

Experiment II

It will be recalled that in this phase of the study, the group which had completed two units of instruction with a machine was shifted to the programmed-text format for the unit on probability; while the group which had worked with the programmed-text format for two units was shifted to the machine. The group which had worked

with conventional text material for the first two topics was shifted to a programmed text with instructions to read the confirming response before writing an answer (this group will be referred to as the "copying group".)

The results are:

(a) Mean time to complete the unit:

machine group	15.7 hours (S.D. 3.1 hrs.)
programmed-text group	13.5 hours (S.D. 3.0 hrs.)
copying group	Time was not recorded

(b) Mean error rate on the unit:

machine group	25.7%
programmed-text group	18.6%
copying group	No error rate possible

Note: $t = 2.001$, $df = 33$, $p > .05$

(t must equal 2.040 for significance)

(c) Analyses of variance on test scores for the three experimental groups:

	F	p
Test I	.451	$> .05$
Test II	.935	$> .05$
Test III	.236	$> .05$

An analysis of covariance on the post-test scores, with the scores on the intelligence test and the Iowa Mathematics Achievement Test serving as control variables, resulted in an F of 13.079 ($p < .01$) for all four groups and an F of 1.265 ($p > .05$) for the three experimental groups indicating (1) that Ss in the experimental groups achieved significantly higher scores on the post-test than

did the controls but, (2) that the differences in achievement among the experimental groups on the post-test of the unit on probability were not statistically significant. Again, as was found on the first unit of instruction, the machine vs programmed text self-study variable did not produce differences in achievement on any part-test, on the post-test, nor, as will be indicated, on the retention test.

An analysis of covariance on the scores from the retention test for the three experimental groups, with the scores from the intelligence test and the Iowa Mathematics Achievement Test serving as control variables, resulted in an F of 2.303 ($p > .05$), indicating that the amount retained was not differentially affected by the mode of instruction. It should be noted that the mean for the machine group was higher than the means for the two groups which used programmed textbooks.

Rank-order correlations between the scores on the post-test and the number of errors committed on the program were significant both for the machine group ($r_s = .615$, $df = 11$, $p < .05$) and for the programmed-text group ($r_s = .536$, $df = 16$, $p < .05$), indicating a tendency for low error rate to be related to high achievement.

The Preference Scale

Following the completion of the unit on Groups and Fields, i.e., at the end of Experiment I, and, again, following the unit, An Introduction to Probability, i.e., at the end of Experiment II, Ss were asked the following questions:

1. Do you believe that you have learned as much from this experience as you would have in a course taught in a

conventional manner?

YES NO

2. Are you bored by the continued use of these materials?

YES NO

3. Would you prefer to use programmed material in another mathematics course, e.g. trigonometry, if it were possible to do so?

YES NO

4. Do you prefer to use a programmed text rather than a conventional text?

YES NO

5. Do you prefer to use a teaching machine rather than a programmed text?

YES NO

6. Which unit of mathematics did you like best?

7. Would you prefer to use the programmed materials with a teacher?

YES NO

8. Would you prefer the programmed material

(a) as a supplement to a conventional text?

(b) as a replacement for a conventional text?

(c) not be used with a conventional text?

A chi-square test of contingency was applied to the frequency data obtained for each question[1], to compare the experimental groups

[1]The test was used only when the theoretical frequency for a class was five or greater.

following the completion of each of the two studies.

No significant differences in response occurred between or among the experimental groups on questions (1) and (2). At the end of the first experiment, i.e. after completing 2300 frames, both the machine group and the programmed-text group reported that they were learning as much as they would have in a conventional course, but, at the completion of the second experiment, Ss tended to shift from the "YES" to the "NO" category, perhaps as a result of the high error rate, and lower achievement, on the third unit. On question 2, the responses of the two groups which worked only with programmed material cluster in the "bored" rather than in the "not bored" category. The third experimental group, which used conventional text material in the first experiment and the learning program in the second experiment was split about evenly in their response to the question on boredom. It is important to restate that no significant differences were found among the experimental groups as a function of the experimental conditions.

The responses to questions (3) and (4) indicate almost unanimous preference for programmed texts rather than for conventional texts and a very marked preference for the use of a learning program in another course.

Question 5, the question concerned with Ss preference for machine versus programmed text produced an interesting set of responses. At the end of the first experiment, the differences between the machine group and the programmed text group were highly significant ($X^2 = 15.293$, $df = 1$, $p < .01$) with the machine group expressing a preference for the machine and the programmed-text group expressing a preference for the mode which it had used.

Following the second experiment in which the groups had changed modes, all three experimental groups now reported a preference for programmed texts although this preference was not strong enough to result in statistical significance.

Almost all Ss preferred the program on Sets, Relations, and Functions to the program on Groups and Fields. When the responses to question (6) following the second experiment are examined, the preference for set theory holds for the original machine group and the conventional text group, both of which had been shifted to the programmed-text format in Experiment II, but the remaining experimental group, a group which had been shifted to machines, changed its first preference to probability theory. It is impossible to determine with confidence whether this shift in preference was directly related to machine presentation. All three experimental groups ranked Groups and Fields as third choice.

From their responses to questions (7) and (8), Ss would prefer strongly to use learning programs as supplements to teachers and to conventional textbooks; moreover, no S reported that he preferred that programs not be used as supplements to or replacements for conventional textbooks. Again, the responses did not differ as a function of the experimental conditions.

V. DISCUSSION

The Primary Hypothesis

The main hypothesis of this study was

> Superior students who use programmed instruction in either machine or text format will achieve significantly higher test scores for the seminar course in topics in modern algebra than will superior students who use conventional textbook material for self-study of the same topics.

Again, it must be noted that the major purpose of the study was to determine the contribution to achievement in mathematics of the combination of a learning program and a weekly seminar; it was not a comparison of programmed instruction versus conventional instruction.

The results which bear directly on this hypothesis are those data reported in Experiment I for the unit on Sets, Relations, and Functions. For this unit, the experimental groups achieved significantly higher scores than the non-exposed controls, simply establishing that the textbook material and the learning programs did teach, in fact. On the other hand, for the experimental groups none of the analyses of variance for the part-tests or the analysis of covariance for the post-test on the unit was significant; as a consequence, the hypothesis is not supported. It must be concluded that the self-study of textbook material combined with a weekly seminar is no less effective than the self-study of learning programs combined with a weekly seminar. Since this treatment entailed the study of over 1000 frames extended over a six-week interval, it would seem to have been of sufficient magnitude to be considered seriously.

Given the many claims for the effectiveness of programmed instruction, this conclusion is discouraging to its proponents. Before

accepting it at face value, it might be well to re-examine the treatment qualitatively. The treatment under consideration was to consist of the variable of the self-study of learning programs versus conventional text material with a weekly seminar common to all groups. The weak point in the treatment was the weekly seminar since each group met as a group with the teacher, thus making it possible for the teacher to provide "more help" to one group than to the others. When each class session was monitored by one of the investigators, the teacher was, in fact, providing more basic information for the textbook group simply because they had more questions which required new information. Thus, it must be concluded that the weekly seminar was not a constant for all groups; hence, the finding of no significant differences in achievement among the experimental groups loses some of its impact. In addition, by tapping the students' "grapevine", it was learned that several Ss in the conventional textbook group had "organized" to meet the challenge. Some of them met for informal "bull sessions" in after-school hours and some purchased books on set theory - later, on group theory, also - in order to be able to learn more about the topic. Even though such procedures confound the independent variable, one can only admire the intelligent responses which some Ss exhibited to a rather frustrating mode of instruction. This experience is a sharp reminder of the difficulty of establishing an uncontaminated independent variable in research on methods of instruction.

After the experience with the seminar for the first unit, the seminar was eliminated for this part of Experiment I, with some rather interesting results. On two of the five part-tests, the analyses of variance resulted in significant Fs. On Test I, both the machine

group and the programmed-text group achieved significantly higher scores than the conventional textbook group, and on Test IV, the machine group scored significantly higher than the conventional textbook group. Further, on all tests except III, the means for the conventional textbook group were lower than those for the programmed instruction groups and the differences approached significance. The analysis of covariance on post-test scores and on the retention test scores also resulted in significant Fs; again with the machine group earning significantly higher scores than the conventional textbook group. The means for the programmed-text group in these two analyses were markedly but not significantly higher than the means for the textbook group.

There were two clear differences between the first half and the second half of Experiment I. No instruction was received from a teacher in the second half of this experiment and the subject matter, Groups and Fields, was judged to be harder by the Ss as well as by mathematicians. Thus, even though the Ss in the conventional textbook group may have engaged in more uncontrolled extra-class study of this topic, without the help of a teacher, they were not able to handle the abstract, complex subject of groups quite as well as did those Ss who learned from programs, and this difference persisted through the retention test for Groups and Fields. One should not lose sight of the fact that the conventional textbook group did learn a significant amount, however. Further, the content of the learning programs and the content of the textbook material were not identical, perhaps handicapping the textbook group somewhat, even though every concept in the criterion measures was presented in the textbook material.

For those readers who are concerned with the role of the teacher in

the use of learning programs in mathematics, the material contained in Appendix C, Extension of the Study, will be of interest.

The Secondary Hypotheses

The place of the machine in programmed instruction still has not been defined clearly. In general, given the type of programs produced thus far, comparative studies of the effectiveness of machine-presented versus text-presented learning programs typically have reported no significant differences in achievement as a function of mode of instruction. (For example, see: Alter & Silverman, 1961; Eigen, 1961; Eigen, 1962; Eigen and Komoski, 1960; Feldhusen and Birt, 1962; Moore and Smith, 1963; Roe, et al., 1960.) Goldstein and Gotkin (1962) recently have reviewed eight such studies, none of which resulted in significant differences in achievement.

Two consistent, but statistically non-significant, differences appeared between the machine group and the programmed-text group in both experiments reported here. The machine group had a higher error rate on each unit and it always used more time in completing each unit; yet, the machine group did not differ significantly in achievement from the programmed-text group on any criterion measure. The type of machine which was used in this study (See Appendix A) was designed to prevent cheating, i.e. to prevent the reading of the confirming response until S had answered the question frame. Presumably, Ss in the machine group were forced to read carefully in order to increase the probability of a correct response. At least this is an assumption which is made frequently by proponents of machine presentation. Ss who used programmed textbooks, although supervised by a proctor, clearly could copy the confirming response without attempting to solve the problem. The fact

that the error rate was consistently lower for this group probably is evidence of a minor amount of cheating by these Ss, although the lower error rate may be attributable to another factor, viz. the opportunity for review of previous frames by the Ss who used programmed textbooks but denied to Ss who used machines. It seems probable that both of these factors, copying and reviewing, contribute to the differences in error rates. A third factor which could be related to the differences in error rate seems not to have been mentioned in the literature. In working through a program in the early stage of its development, some proportion of the frames or some sequences of frames can be assumed to be so poorly constructed as to be very difficult for a student. When used in a machine, programs of this quality must be perceived by some Ss as a source of frustration. If so, some Ss would respond by avoidance behavior, i.e. they would cease to be overly concerned about the correctness of their responses; instead, they would concentrate on the reading of the teaching frame, respond carelessly or inappropriately when answering, and read the confirming response carefully. Since reading the confirming responses before answering seems to be as effective as the conventional procedure of responding, a fact which S could discover after his first examination in this study, S loses little by careless or irrelevant responses. If only a few Ss - five to ten per cent - adapt in this manner, the error rate for a machine group will be higher than that for a programmed text group, assuming that some Ss in a programmed-text group take advantage of the opportunity denied to the machine group; viz., the opportunity to review previous frames and the opportunity to copy confirming responses.

It is apparent from the data presented in this study that the programmed material was in need of improvement since error rates varied from a low of 13.4 per cent for the programmed-text group on Sets, Relations, and Functions to a high of 25.7 per cent for the machine group on An Introduction to Probability. The error rate for the programmed-text group also was highest on the unit on probability, serving as a reminder that error rates seemingly vary as a function of difficulty for the programmed-text group as well as for the machine group. The machine group in Experiment II perhaps found the use of machines more frustrating than the machine group in Experiment I since the former had four months' experience with the flexible programmed texts before they began work with the machine. The responses of all Ss to question (5) on the preference scale provide support for the conjecture that the contribution of machines seemingly is perceived as negative by Ss who have had experience with both machines and texts.

Returning to the hypothesis that some Ss respond to the machine mode with avoidance behavior we might note that the answer tapes for some Ss in the machine group differed in two respects from the answer tapes for any S in a programmed-text group. First, the rate of non-response was practically zero for the programmed-text group but, at certain points, it was relatively high for the machine group. Second, irrelevant responses never appeared on any answer tape for the programmed-text group but, at certain points, such responses did occur for the machine group. The irrelevant responses usually consisted of nonsense expressions, attempts at humor, "poetry" and rhymes, or direct expressions of hostility, e.g. "I think this is too much!"

"This stuff bugs me!", "I'm not very bright, but this is stupid", and "This is not well-written." Perhaps these two observations can be considered to provide indirect support for the hypothesis outlined above.

When these results are added to those reported in the several studies in which machine mode and programmed text have been compared it is evident that (1) machines do not automatically contribute to achievement, (2) cheating or copying of confirmation frames before or instead of first producing an answer seems not to affect achievement, unless the cheating results in greatly reduced covert responding, (3) machines sometimes, but not always, require Ss to spend significantly more time on the same amount of material, and (4) machines may contribute more to a "pall effect" than do programmed textbooks. Seemingly, unless a low error rate is made to be important to a student and, in fact, is of importance in increasing achievement and/or retention, it is difficult to conceive of any need for a machine which prevents cheating. Eigen (1961) and Eigen and Komoski (1960), Komoski (1962), and Goldstein and Gotkin (1962), among others have argued this point with cogency. Until the correlations between achievement and/or retention on criterion measures and error rates on programs are markedly higher than those found in this and other studies, there is little merit in devoting further attention to trying to bring Ss to perceive their error rates as crucially and invariably related to their achievement.[1]

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[1]There is no intention of suggesting that the error rate on a program is meaningless information to the consumer of programs; rather, it is suggested that the relationship between error rate and achievement is not of sufficient magnitude to predict achievement from error rate for any practical purposes.

The Preference Scale

The results reported for this scale require little interpretation or discussion. In the main, they agree with the attitudes and preferences expressed to the same questions by college students (Moore and Smith, 1963). Students believe that they learn as well with programmed material as they do in a "traditional"course; they like programmed material, particularly as a supplement to other instructional materials and teacher-centered instruction; they prefer a programmed textbook to a machine. They become bored with sustained exposure to learning programs. Let no one forget that students also become bored with sustained exposure to lectures, to films, to textbooks, etc.

The most rewarding expressions of attitudes and preferences, naturally enough, came from students, teachers, and parents directly rather than from the tabulated "yeses" and "noes". For a few students, their first successful experiences with, and liking for, mathematics occurred when they discovered learning programs. A science of instruction cannot be built on anecdotes but some scientists engaged in research on instruction frequently find that their motivation to continue in the face of one "no significant difference" after another is helped by an occasional anecdote, particularly when reported by a child who suddenly is enthusiastic about and successful in a subject which heretofore has been avoided and disliked. The report is even more rewarding when one discovers that a student has purchased programmed material for further study or, in one instance, has purchased a programmed course in calculus as a Christmas present for a beloved father whose education was terminated by family responsibilities. Many, many reports of greatly improved achievement in mathematics were received from students during the months since the study and the later field-testing of revisions of

the three programmed units were completed. Although these clinical observations in themselves do not provide the kind of evidence necessary for obtaining specific conclusions concerning the values of programmed instruction, they do suggest that programmed instruction may be most meaningful when optimal conditions for learning are evaluated in terms of the individual rather than in terms of a group.

Programming Technology: The LTOR method for the revision and preparation of programmed instructional materials in mathematics.

Whatever the differences of opinion may be as related to the factors which contribute to the effectiveness of programmed instruction, most agree that careful empirical testing and revision of the program during development make a significant contribution to its ultimate effectiveness. This is not to suggest that the emphasis which has been placed on the techniques of frame writing is unwarranted; rather that too little attention has been given to procedures used in evaluation. Specifically, in the revision of a program, a programmer may be very sophisticated in his ability to apply the most desirable techniques in the construction of the frames required, but if he does not have information concerning the specific source of learner difficulty within the program, he cannot make the necessary revisions effectively. As a result, many revisions are prepared not only inefficiently but, as contradictory as it may seem, they may make the program somewhat less effective in certain respects even though the overall effectiveness may be increased.

Because of the importance of empirical testing of programmed

material, the authors gave thoughtful consideration to current methods of evaluation. The methods used most commonly were two. First, a determination of the learner's error rate on the program, and second, an evaluation of learner's achievement through the use of appropriate tests. The value of the achievement test in evaluating the worth of a program was supported, but the learner's "error rate" proved to provide not only a poor estimate of the effectiveness of a program, but a poor indication of the source of problems within the structure of the program, also. For example, if there were a high error rate for a specific frame, it was difficult to decide in what manner the frame was related to the final achievement of the learner, e.g., was it a function of practice, a crucial step in the acquisition of a concept, or merely an ambiguity which contributed little to the acquisition of the concept? A further problem, and one not unrelated to the first, is that current methods of evaluating learning programs provide little information concerning necessary or unnecessary redundancy in a program. Since unnecessary redundancy has been one of the chief criticisms of many programs, this problem was considered to be of some significance.

Recognizing that these problems could be overcome only by a better definition of programming, a definition which would provide greater precision in evaluating specific frames, an analysis was made of the functions of specific frames within a program. All frames within the structure of a program, other than criterion frames, were classified as learning or overlearning frames. Learning was defined as the ability of an individual to respond correctly to a stimulus frame; overlearning was defined as meaningful practice of the learned response. In actual practice, the frames within any program were

classified into four general types:

1. Teaching frames (learning frames): these frames are ones which introduce new information necessary for the learning of a concept.
2. Type I criterion frames: non-cued frames included for the purpose of evaluating the learning of the content of the teaching frames in the sequence.
3. Practice frames (overlearning frames): frames which follow Type I criterion frames and are similar in difficulty and content to the Type I criterion frame; they are included to assure retention of the initial learning.
4. Type II criterion frames: frames, similar in content and level of difficulty to the Type I criterion frames, but which are used to determine whether a sufficient number of practice frames have been included in the program.

The sequencing of the four types of frames in a program is as follows: (1) A concept is introduced by a series of teaching frames which is followed by a Type I criterion frame. (2) Practice frames are inserted after the criterion frames which are followed by teaching frames for a new concept, which in turn are followed by a Type I criterion frame, etc. (3) At the end of a logical unit of instruction, one Type II criterion frame is included for each Type I criterion frame in the unit.

This procedure which the authors have labeled as LTOR[1] makes possible the following analysis: First, if the student answers the

[1] L.T.O.R. (pronounced el'tor) is an abbreviation for learn, test, overlearn, and retest.

Type I criterion frame correctly, even though he may have made some errors on the teaching frames preceding it, the programmer can be reasonably certain that the student has learned. On the other hand, if the student answers the Type I criterion frame incorrectly, but makes no errors on the teaching frames preceding it, the effectiveness of the teaching frames is doubtful. Second, if the answer to the Type I criterion frame is correct and the answer to the Type II criterion frame is incorrect, it is evident that learning has occurred, but there probably are an insufficient number of practice or overlearning frames in the sequence. Thus, this method provides a systematic method of evaluating specific frames for revising the program by specifying whether an error is a function of the teaching frames or of the practice frames.

Since the method of evaluation described is important mainly as a diagnostic tool for classifying the empirical data for the programmer, considerable care must be given to the procedures followed in testing the sequences within a program.

The typical testing program used now by the writers is:

1. As soon as a small number of frames (50-75) have been prepared by a programmer, a test-student judged to be typical of the population for which the material is being prepared, responds to each frame in the presence of the programmer. When the student cannot respond correctly or does not understand the response he has given, the programmer attempts to ascertain what the nature of the difficulty is. A record is made of the errors and the correction suggested, and additional frames are added as required. Particular attention is paid to the student's responses to the criterion frames.

2. A second student, who has met the prerequisites for learning the concepts included in the unit and who is from the lower achievement level of the population for which the material is being developed, completes the revised form of the program in a manner similar to that described for the first test-student.
3. When the program has been revised again, a student of low average ability completes the program.

Each of the test-students is required to take an achievement test for each unit of the program, and this data, with the data from the completed units, is then analyzed. An analysis is made of the errors on the achievement tests and these data are compared to the error rates on the comparable Type II criterion frames.

4. After the third test-student has completed the learning program, it is revised again and the entire unit is edited, typed, and reproduced. At this point, a test population is selected for group testing. The criteria used in selecting the test group are two: (a) the group must be representative in educational preparation and intellectual ability of the population of learners for whom the programmed unit was prepared and (b) the learning program must fit into the sequence of the regular curriculum which the students are taking. All students of the class or classes selected are pre-tested with an achievement test constructed to determine their knowledge of the topics included in the objectives of the program. In order to insure that the data resulting from the field testing will be as useful as possible, each classroom teacher is requested to distribute the programmed material to the students at the beginning of each class period, to proctor the group as they work on the individual lessons, and to collect all

materials at the end of each class. Further, the teacher is asked to provide the student with no assistance in answering any of the frames at any time. The teachers and the students are provided with specific instructions for the use of the materials at the beginning of the testing program; thus, no serious administrative problems have occurred. Since motivation to learn is an important variable, all students are informed that their grades will depend, in part, on their achievement.

5. When the field test is completed, all programs and unit tests are scored and the error rate for each frame and test item is tabulated. The data then are evaluated as described previously in this report. When the error rate exceeds ten per cent for any Type I criterion frame, the preceding teaching frames are revised. The Type II criterion frames are treated similarly with additional practice frames included when the error rate exceeds ten per cent. An exception to this procedure is the case in which the student misses both the Type I and the Type II criterion frame for a concept. When this occurs it is assumed that the learning frames are inadequate and revisions in them are made. Unit test items are compared to the Type II criterion frames and when serious discrepancies occur additional practice frames are included.
6. When this revision is completed, the program is edited, typed and reproduced for use in a final field-test.

Although the LTOR evaluating procedure was prepared originally to aid programmers in revising on instructional units, it has, in addition, proved to be extremely useful to teachers and students in the use of programmed materials. For example, a teacher who is thoroughly versed in the material his students are learning through the use of a programmed

instructional unit and who may wish to correlate a lecture with the material, can identify quickly the specific objectives of the assigned unit by examining the Type II criterion frames at the end of each unit. Since there is at least one Type II criterion frame for each Type I criterion frame in each unit, he can be sure that the Type II criterion frames are representative of all of the concepts presented in the unit. This is of particular importance to competent teachers, who would gain little from working through the program step by step but who desire a comprehensive knowledge of what is included in the unit.

Further, the organization of programmed materials in this form is of particular value for students when they are using a program for review, for teachers who wish to assist students in doing remedial work, and for the learner who may have a knowledge of some of the concepts presented in the program but not a complete knowledge of all of the concepts included in the program. Specifically, the learner in any of these groups can make the most effective and efficient use of the programmed materials which follow the form described in this report by adhering to the following procedure:

1. Respond to all of the Type II criterion frames that he encounters at the end of the unit.
2. For any Type II criterion frame which he misses and/or doesn't understand he would refer directly to the comparable Type I criterion frame in the unit. (Since all criterion frames are marked, locating them is a simple matter.)
3. The learning frames immediately preceding the Type I criterion frame would then be reviewed.

It is clear that programmed instructional units organized in this

form make it possible for students to diagnose their weaknesses quickly and systematically, and to overcome these difficulties without devoting time to the review of concepts learned previously.

In summary, the LTOR system of evaluation not only increases the quality and the efficiency of program construction; it also provides a more effective and efficient instrument of instruction for both teacher and learner.

A General Conclusion

Within the very real limits of this study, it can be concluded that self-instructional materials in mathematics can be used profitably by superior students with or without a teacher and/or a traditional text. Our experience from field-testing programs of good quality (See Appendix C) supports this conclusion strongly. We are equally certain that the importance of the able teacher has not been reduced by the introduction of learning programs. It is particularly important to have a teacher whose attitudes toward the subject matter and the media of instruction are positive and that the students perceive them as positive. One would take this for granted in the traditional modes of instruction; it holds, also, for the "new media". The patterns of utilizing learning programs are many. The specific role of the teacher or the conventional text can and must be varied to fit the pattern which is suited best to a specific instructional situation. Considerable research and development will have to be done before the most suitable patterns for specific types of situations are found.

On the basis of our experience in field-testing programmed units of instruction in many settings, several ways in which a programmed

textbook in mathematics might be used in the schools are suggested in Table I.[1] A few comments on each division of the table may be

TABLE I

Programmed Text Used	Condition	Activity
I. For Self-study		
A. With teacher	(1) In class	Self-study supplemented by lectures and discussions
	(2) Outside class	Tutorials, informal discussions at irregular intervals.
B. Without regular teacher	(1) In class	Self-study with teacher available for occasional assistance
	(2) Outside class	Self-study with teacher available for occasional conference
II. As Supplement		
A. With teacher	(1) In class	Self-study correlated with other course materials; lectures and discussions
	(2) Outside class	Self-study correlated with other course materials; supplemented by discussion with the teacher
B. Without regular teacher	(1) In class	Self-study correlated with other course materials; supplemented by occasional assistance from a teacher
	(2) Outside class	Self-study correlated with other course materials; supplemented by occasional conference with a teacher

[1]This material has been taken from a teacher's manual for learning programs prepared by J. William Moore and Wendell I. Smith, to be published by McGraw-Hill Book Co., 1963.

helpful. If a programmed text is used for self-study, it is assumed that it will carry the primary responsibility for teaching its own contents.

If a full course is constructed around self-instructional materials, there is an option of giving the course with or without a regular teacher of mathematics and of using the programmed text material either in or outside of class. In the field testing of programs, teachers have tried all four of these variations with success. Without hesitation, it must be stressed that success in using the programmed material is more closely related to the teacher's attitude toward, rather than to the teacher's competence in, mathematics in programmed form. Students enrolled in courses with teachers who had no previous knowledge of or experience with the topics covered in the programs, but whose positive attitudes toward programmed self-instruction and modern mathematics were communicated to the students, tended to achieve on a significantly higher level than students enrolled in courses with teachers whose knowledge of modern mathematics was greater, but whose attitudes toward the programs and/or their content were essentially negative. It seems essential then, that programmed material, like any other successful teaching material, have the teacher's support. Any one of the four variations of self-study suggested in Table I will be effective if the study is done for or under the guidance of a teacher who can communicate positive attitudes toward the material to the student.

In the field testing, some teachers had the students work on the material only in class, and the teacher's role was limited to providing help to those students who requested it. In addition, some teachers spent a part of each period or one period a week in discussing the programmed material and relating it to previous work in mathematics, or

in introducing additional applications of the concepts. In small schools and in providing enrichment for gifted students, it is possible to have several students meet regularly with one teacher with the students working on several different courses, each student working at his own rate on course material appropriate to his training, his interests, and his needs.

Programmed texts may be used as a supplement to courses in mathematics in which either a "traditional" or a "modern" text is used. Again, it may be assigned for study in class or outside of class at the option of the teacher. Since it is a self-teaching text, it can be used in a much greater range of settings than can a conventionally written book.

By having students record their answers and having them check those which they feel they do not understand, the teacher obtains a record on each student's progress and, in addition, receives information of great value to him in preparing for discussions of the material with the students.

A highly useful article which might be "required reading" for anyone who chooses to undertake research of the type described in this report is "Teaching Machines or What?", a discussion of some of the pitfalls and cautions of comparative studies of machines, textbooks, and teachers. (Matchett, 1962). Matchett concludes her article with,

> "....the textbook suffers from a bad reputation at the moment. There is a general predilection for technology, and the printing press is not regarded as a very advanced technological achievement. Machines motivate partly because they are machines; books can motivate at most by what they contain. But the question of content goes to the heart of education, and an inquiry which emphasizes it can scarcely fail to have value." (p. 355)

APPENDIX A

THE TEACHING MACHINE

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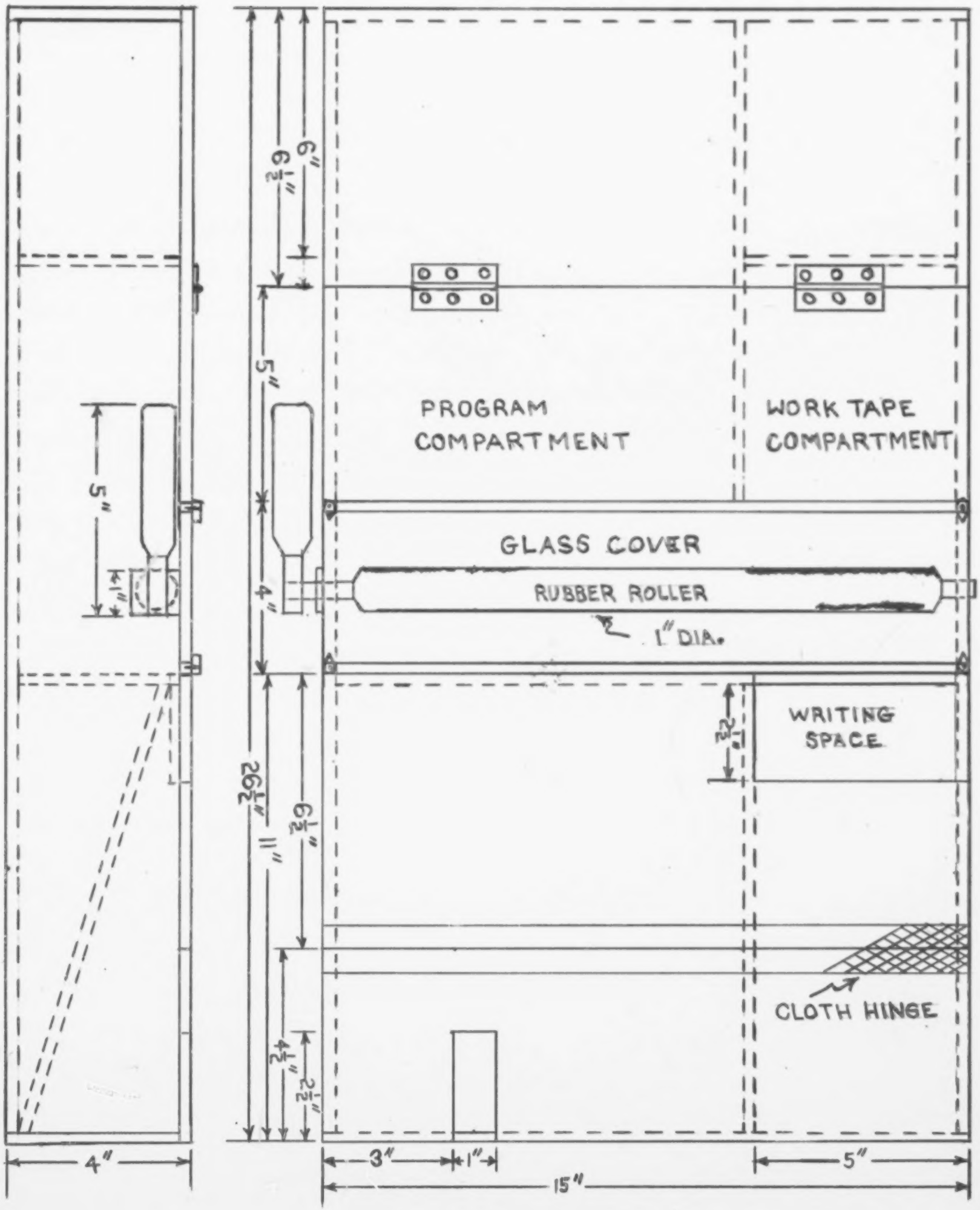
Because of the amount of labor and material required in preparing rolls of programmed material for use in the most satisfactory machine then available, the writers designed a machine in which it was possible to use $8\frac{1}{2}$ x 11 multilithed coverstock material. A schematic diagram of this machine is given on the next page.

The only requirements which the machine had to meet were

1. It had to feed separate sheets of $8\frac{1}{2}$ x 11 coverstock material past an aperture one frame, and only one frame, at a time.
2. It had to be reasonably cheatproof and prevent vicarious review; therefore, it had to have the capacity to store enough new material for one hour's work and it had to store each sheet as the student finished it; in both cases, the storage compartments could not be visible to the student.
3. It had to feed an answer tape which was synchronized in movement with the movement of each frame in order to prevent the student from (a) reading the confirming response until he had responded to the frame, (b) from changing his answer to any frame, and (c) copying any preceding response as a response for a frame now requiring an answer.
4. It had to be easy to operate, simple to maintain, easy to store, and inexpensive to construct.

The final version of the machine which was used met all of these requirements extremely well. Essentially, the machine is a flat box



TEACHING MACHINE

15 x 26$\frac{1}{2}$ x 4 inches consisting of (a) a compartment in which the program is placed face-up and a second compartment into which each sheet falls as it is used, (b) a rubber platen in contact with a variable pressure glass plate under which each page of the program moves as the platen is turned, (c) a compartment in which the answer tape is placed and a second compartment into which the expended tape falls, and (d) a writing surface under a portion of the tape on which the student records his answer. The program is fed onto the platen sheet by sheet simply by inserting the eraser-end of a pencil into a slot in the cover of the compartment closer to the student.

Students had no problems in operating the machines; in fact, in the several months during which the machines were used, not one had to be repaired, an experience which we have not had with any commercially-produced machine. The chief disadvantage of the machines would appear to be their bulk, hence, an excessive demand for storage space. This problem was solved simply by stacking the machines fifteen to the pile. Given a cost of less than fifty dollars per unit, a machine of this type seems to be as satisfactory as any known to the authors.

APPENDIX B

Table of Contents

1. Sets, Relations and Functions

I	Sets, description and notation
II	Operations with sets
III	Equations and inequalities
IV	Ordered Pairs
V	Cartesian products
VI	Locus
VII	Relations
VIII	Functions
IX	Review

2. Groups and Fields

I Binary Relations

1. Ordered Pairs
2. Cartesian Products
3. Binary Relation
4. Properties of a Binary Relation
5. Equivalence Relation
6. Partition of a Set
7. Equivalence Classes
8. Review

II Binary Operations on a Set

1. Function and Mapping
2. Permutation of Degree N
3. Permutation Multiplication
4. Binary Operation on a Set
5. Properties of a Binary Operation

III Groups

1. Symmetries of an Equilateral Triangle
2. Group
3. Proofs of Some Theorems on Groups
4. Mappings and Transformations
5. Transformation Group

IV Appendix to Groups

An Example of a binary operation which is not well defined with respect to a given equivalence relation.

V Fields

1. Review of Binary Relations and Properties
2. Review of Binary Operations and Properties
3. Example of Sets with 2 Binary Operations
4. Fields
5. Proofs of Some Theorems on Fields

3. An Introduction to Probability

I Frequency and Relative Frequency

II Sample spaces, Events, and Preliminary Notions about Probability

III Complementary Events and Review

IV Combinations of Events and Computation of their Probabilities

1. Intersection of Events and Conditioned Probabilities
2. Dependent and Independent Events
3. Review and Further Discussion of Sample Spaces, with Emphasis on Independent Events
4. Union of Events
5. Review

V Binomial Distribution, Permutations and Combinations

VI Review

Sample Frames

1. Sets, Relations and Functions

Q1.1
A set is a collection of things. A collection of paintings in a museum, a flock of wild geese, or the natural numbers from 1 to 15 are examples of _____ of things.

A1.1

sets

Q1.2
We think of a set as a collection of objects: books, trees, numbers, points, etc. Thus, you could say that the collection of books you now have is a ____ of books; Mr. Brown's orchard is a ____ of trees; 1, 2, 3, 4 is a _____ of numbers.

A1.2

set, set, set

Q1.3
The individual members of the set are called its elements. As an example, consider the set consisting of the four boys whose names are Bill, John, Joe, Tom. This set has four members; each one is an _____ of the set.

A1.3

element

Q1.4

Sets may have few or many elements (or members). (We use interchangeably "element", "member"; "is a member of", "is an element of".)

(1) How many elements has the set consisting of the integers 1, 2, 3, 4, 5?

(2) How many members has the set which we call our alphabet?

A1.4

(1) 5, (2) 26

Q1.5

A set may have a finite or an infinite number of elements. Classify the following as finite or infinite sets.

(1) the set of months in the year

(2) the set of Chicago telephone subscribers listed in the Chicago telephone directory

(3) the set of all positive integers

Note, intuitively a set is called <u>finite</u> if the number (large or small) of elements can be counted (given the time and the opportunity). Examples are apples in a basket, grains of rice in a box. If the counting process would be endless, we say the set is <u>infinite</u>. An example is the set of all points on a line.

A1.5

(1) finite, (2) finite, (3) infinite

Q1.6

If we start counting the elements of a set, one of two possibilities may occur.

(1) A definite number is reached beyond which no more are needed because each element has already been counted; or

(2) no such number is reached because there are always elements of the set that have not been previously counted.

Label (1) and (2) as finite or infinite sets.

A1.6

(1) finite, (2) infinite

Q1.7

The positive integers less than 100 which may be written $\{1, 2, \ldots, 99\}$ is an example of _____ set. This set has exactly 99 elements.

Use finite or infinite for the blank above.

A1.7

finite

Q1.8

The set of positive integers greater than 100 which may be indicated $\{101, 102, \ldots\}$ is an example of a set which we would call _____.

A1.8

infinite

Q1.9
If we make a list to show the elements of a set, the order of listing is not important. For example, the set $\{a,b,c,d\}$ is identical with the set $\{c,b,a,d\}$. Is the set whose elements are 0,2,1 the same as the set whose elements are listed 0,1,2?

A1.9

yes

Q1.10
Moreover, if we list the elements of a set, we do not "count a member more than once". The members may be listed $\{0,1,2,1,0\}$; but there are just three elements of that set, namely 0,1, and 2.
How many elements are in the set $\{2,1,2,1,3,1,3\}$?
List them.

A1.10

three 1,2,3

Sample Frames

2. Groups and Fields

Q2.752
Once again, we are in a position of finding an underlying theme which encompasses a great number of seemingly dissimilar structures. In all of these past problems we have found a set having a binary operation which is associative, for which there is an identity element, and such that each element has an inverse. Therefore, it will be profitable for us to study such a structure in general; for all the examples we have seen are simply particular cases; and whatever we find about the structure in general will be true for the particular cases. In studying the more general case we will gain a greater insight whenever we see a particular example.

No answer required.

A2.752

No answer required.

Q2.753
Definition: given a set S having a binary operation *, S is a group with respect to that binary operation if:
(a) the binary operation is associative in S,
(b) there exists an identity element in S,
(c) every element of S has an inverse in S.
For example, let S be the set of integers with respect to addition.
(1) Addition is a binary operation in S.
(2) Addition of integers is associative.
(3) There is an identity element in S (0).
(4) Every element of S has an inverse. (The inverse of a is (-a).) Therefore, the set of integers is a _____ with respect to _____.

A2.753

group, addition

Q2.754

The set S of integers is not a group with respect to multiplication since one of the properties is not satisfied.

(a) Is multiplication a binary operation in S?

(b) Is multiplication in S associative?

(c) Is there an identity element? If so, what is it?

(d) Does every element of S have an inverse in S?

A2.754

(a) yes, (b) yes, (c) yes, 1, (d) No, for example, find the inverse of 5.

Q2.755

Remember, the inverse of an element a under a binary operation is that element which, operating on a, will give you back the identity element. In the last item, the integers under multiplication, the identity element is 1. Now note that, for example, 5 has no inverse in S. There is no integer x such that $5 \cdot x = x \cdot 5 = 1$. Therefore, the set of integers under multiplication (is or is not) a group.

A2.755

is not

Q2.756

Maybe we can patch the last set up so that it will be a group with respect to multiplication by including the fractions we need for inverses. So, let S be the set of rational numbers under multiplication.

(a) Is multiplication a binary operation?

(b) Is multiplication of rational numbers associative?

(c) Is there an identity element ϵ S?

(d) Does every element of S have an inverse in S?

A2.756

(a) yes, (b) yes, (c) yes (1), (d) No, 0 does not have an inverse.

Q2.757

If S is the set of rational numbers under multiplication, the identity element is 1. $0 \epsilon S$, but there is no $x \epsilon S$ such that $0 \cdot x = x \cdot 0 = 1$. Every other element of S does have an inverse; but still, the property demanding that every element of S have an inverse is not satisfied. Therefore, the set of rational numbers (<u>is or is not</u>) a group with respect to multiplication.

A2.757

is not

Q2.758

Suppose we get rid of the one element which gave us trouble. Let S be the set of rational numbers excluding 0.

(a) Is multiplication a binary operation in S?

(b) Is multiplication associative in S?

(c) Is there an identity element in S?

(d) Does every element of S have an inverse in S?

A2.758

(a) yes, (b) yes, (c) yes, (d) yes

Q2.759

From the last item, then, the set of rational numbers excluding zero is a _____ with respect to multiplication.

A2.759

group

Q2.760

The definition of a group can be written symbolically as follows: if S is a set of elements having a binary operation * such that,

(1) for all $a,b,c \in S$, $a*(b*c) =$ _____,

(2) there exists an identity element $e \in S$ such that for all $a \in S$, $a*e = e*a =$ _____,

(3) for each $a \in S$, there exists an inverse a^{-1} such that ___ = ___ = ___.

A2.760

(1) $(a*b)*c$

(2) a

(3) $a*a^{-1} = a^{-1}*a = e$

Q2.761

A <u>group</u> is so important that you should memorize the properties; and, actually, they are quite simple. A set S having a binary operation which is associative and having an identity element with respect to that binary operation and such that every element of S has an inverse in S is called a _____.

A2.761

group with respect to that binary operation

Q2.762

Let S be a set having a binary operation ($\circ$) such that,

(1) for all a, b, c, $\in$ S, $(a\circ b)\circ c = a\circ(b\circ c)$ (associative),

(2) there exists an element a $\in$ S such that for all a $\in$ S, $a\circ e = e\circ a = a$ (identity),

(3) for all a $\in$ S, there exists $a^{-1} \in$ S such that $a\circ a^{-1} = a^{-1}\circ a = e$ (inverse).

Then, S is a _____.

A2.762

group with respect to the binary operation $\circ$

Q2.763

In order for a set S to be a group it must satisfy all the group properties (binary operation, associative, identity element, and inverse). Is the set of even integers a group with respect to addition? If not, what property is not satisfied?

A2.763

yes

Sample Frames

3. An Introduction to Probability

Part I - FREQUENCY AND RELATIVE FREQUENCY

Q4.1

A light-bulk manufacturer performs the following experiment: From a set of new bulbs, one bulb is selected and inserted in a light socket which is turned on. If the bulb lights, a G (for good) is recorded. If the bulb does not light, a D (for defective) is recorded.
Suppose this experiment has been performed for 10 different bulbs and the following result recorded: GGGDGGGGGD.
If n(G) (read n of G) represents the number of good bulbs among the 10, n(G) = 8.
If n(D) (read _____) represents the number of defective bulbs among the 10, n(D) = ____.

A4.1

n of D 2

Q4.2

A biologist performs the following experiment:
A healthy mouse is injected with a vaccine and the exposed to disease X. If the mouse does not contract disease X, a W (for well) is recorded. If the mouse does contract disease X, an S (for sick) is recorded.
Suppose this experiment has been performed in the same manner on 10 mice and the following results recorded: SWWWSSWWWW.
If n(S) represents the number of mice of the 10 contracting disease S, n(S) = _____.
If n(W) represents the number of mice of the 10 not contracting the disease, n(W) = _____.

A4.2

3 7

Q4.3

Consider the following experiment:
A hat contains a number of balls; some red, some white, some blue, some yellow. A single ball is drawn from the hat, its color observed, and the ball is replaced. Let R,W,B, and Y represent the colors.
Suppose this experiment has been performed 12 times and the following recorded: RRYRWWRWWYRR
If n(K) is the number of balls of the 12 drawn having color K, then n(R) = ___ n(W) = ___ n(B) = ___ n(Y) = ___

A4.3

6 4 0 2

Q4.4

Consider the following experiment:
A coin is tossed in the air and allowed to come to rest. Observe whether a head (H) or a tail (T) appears.
Suppose this experiment has been performed 10 times with the following results: HHTHTTHTHH
If n(H) is the number of times out of the 10 tosses which a head has appeared, then n(H) = ___.
Similarly, n(T) = _____.

A4.4

6 4

Q4.5

In an experiment of tossing a coin, we agree if anything unusual happens, (such as the coin landing on edge, or breaking in half, or disappearing), we will disregard the result.

Suppose a coin has been tossed 10 times and the following recorded: TTHTHTTTTH

Then n(H) = ___

n(T) = ___

A4.5

3 7

Q4.6

Consider the following experiment:

Joe Zilch runs the 100 yard dash and his time (in seconds) is recorded.

Suppose, during the last week, this experiment has been performed 7 times with the following results:

10.7, 10.3, 11.0, 11.1, 10.2, 10.0, 10.3

Let A be the event that Joe's time for the hundred yard dash is less than 11 seconds.

If n(A) is the number of times (out of the 7 trials) which event A occurred, then n(A) = ___.

If B is the event that Joe's time is less than 10.8 seconds and greater than 10.1 seconds and n(B) is the number of times in which event B has happened, n(B) = ____.

A4.6

5 4

Q4.7

In all of the preceding examples, we considered a certain experiment. The experiment was performed N times, and the results observed and recorded.
We then singled out some event A and counted the number of times event A occurred and indicated this number by n(A).
For example, a coin is tossed 11 times with the following results: HHTHHHTTHTH
N = ___ n(H) = ____ n(T) = ____

A4.7

11 7 4

Q4.8

Suppose an experiment has been performed N times and the results recorded. Let A be some specified event, then out of the N trials of the experiment, A has occurred a certain number of times. Note that the number of times which A has occurred is indicated by n(A). (read n of A). n(A) does not indicate any sort of multiplication. The following is the result of tossing a coin 8 times: HTTHTTTH
N = ___ n(H) = ___ n(T) = ___

A4.8

8 3 5

Q4.9

If an experiment is performed N times and out of these N results, event A has occurred n(A) times, the relative frequency of event A = $\frac{n(A)}{N}$

For example, out of the 10 babies born in the local hospital yesterday, the following results were recorded where G represents girl and B represents boy. BBGGBGGGBG
If G is the event that a girl is born, n(G) = ___
and the relative frequency of event G = $\frac{n(G)}{N}$ = ___.

A4.9

6 $\frac{6}{10}$ (or .6 or $\frac{3}{5}$)

Q4.10

If an experiment is performed N times and event A occurs n(A) times, then the relative frequency of event A = $\frac{n(A)}{N}$.

The relative frequency of event A, then, is the ratio of the number of times event A occurs to the number of times event A is given the opportunity to occur.
Suppose the weights of 10 girl students from your school are as follows: 110, 150, 96, 112, 138, 116, 120, 100, 110, 115.
If A is the event that a weight is less than 120, then, for this experiment, N = ___, n(A) = ___,
relative frequency of A = ___.

A4.10

10 7 $\frac{7}{10}$

Q4.11

The notation we have been using is as follows:
The number of times an experiment has been performed is indicated by _____.
The number of times event A has occurred is indicated by _____.
The relative frequency of event A is given by _____.

A4.11

N $n(A)$ $\frac{n(A)}{N}$

APPENDIX C

EXTENSION OF THE STUDY

In 1961, non-federal funds were secured for the purposes of (a) revising completely the learning programs used in this study and (b) field-testing learning programs in several schools under a variety of learning situations. This Appendix will consist of a summary of some of the field test data for one of the three learning programs, Sets, Relations and Functions.

Sets, Relations and Functions

1. Student volunteers from rural high schools

   Twenty high school juniors and seniors met once each week for two hours on a college campus. The students worked on programmed material both in and out of class. The teacher answered questions each week and discussed any points which were unclear to students. The students had had only two years of high school mathematics and they had had no previous training in set theory. The mean score for this group on the final examination was 93 per cent and the mean error rate on the program was four per cent.

2. High school teachers of mathematics

   Fifteen teachers of secondary school mathematics enrolled in a summer program for in-service training in modern mathematics worked through the program during a weekly laboratory period. No discussions of the material were held and no other form of teacher participation occurred. None of the teachers had had any training in set theory. The mean score on the final examination was 88 per cent and the mean error rate on the program was six per cent. Two students scored below 80 per cent on the final examination.

3. Urban high school students

This population consisted of forty students enrolled in a twelfth grade course in "modern mathematics". Assignments in the program were made daily for extra-class study and fifteen minutes of each class period was devoted to a discussion of the previous assignment. The program was correlated with text material as well as with teacher-led class activities. No previous training in set theory had been given. The mean score on the final examination was 90 per cent and the mean error rate on the program was only two per cent. Only three students scored below 80 per cent on the final examination.

4. College freshmen enrolled in education

One hundred students enrolled in a required course in freshman mathematics for majors in education worked through the program during class hours under the supervision of a teacher and two graduate assistants. For a sample of thirty, the mean score on the final examination was 90 per cent and the mean error rate on the program was five per cent. The variability in this population was much greater than for the preceding three since nearly twenty-five per cent of the group scored below 80 per cent.

5. Evening students enrolled in a junior college in a "traditional" algebra course

This group of twenty-three students were assigned the program for extra-class work. The material was never discussed in class nor was it related to the content of the algebra course in any way. Twelve of the students

failed to complete the program at all and of the eleven who did, only eight exceeded a score of 80 per cent on the final examination. It was impossible to determine the mean error rate for this group since few of them recorded their responses to the frames.

The performance of this group serves as a reminder, again, that learning programs when misused will not produce results comparable to those which are reported for a program properly used by a teacher or used under the conditions specified by the program author.

Altogether, Sets, Relations and Functions was tested on twelve separate populations totaling more than three hundred students. Some of the generalizations which stem from the extensive field testing of this and other programs in the schools are:

1. Only those teachers who are willing to experiment with instructional materials and who develop and execute a plan for their use are likely to find that their students' error rates will be low and their achievement will range from good to excellent.
2. The highest achievement probably will occur for those groups with whom a teacher works closely, particularly one who correlates class discussions with the assignments made in the learning program.
3. A learning program should be used in a manner and under conditions which will lead to students' perceptions of it as natural or normal instructional material, rather than as a gimmick or a curiosity.
4. Teacher attitude toward programmed instruction probably is the most important single variable in determining the success

of a carefully developed, well-tested program.

5. Learning programs in mathematics seemingly can be used with marked success in more educational settings with less teacher investment than can any other instructional material developed to this time, given the conditions and limitations stated in the first three generalizations.

APPENDIX D

Criterion Measures

Achievement Test for Sets, Relations, and Functions

1. Write the following in words:
   (a) $A \subset B$ (b) $a \in S$ (c) ϕ (d) $A \cup B$
   (e) $A \cap B$
2. If $J = \{x \mid x \text{ is an integer}\}$
   (a) is 2 an element of J?
   (b) is -7 a member of J?
   (c) is $\sqrt{3}$ an element of J?
   (d) is $\{-5, 1/2, 1, 4\}$ a subset of J?

3. Let $A = \{2,4\}$ and $B = \{1,7\}$ what is A x B?

4. Make a Venn diagram showing that sets A and B are subsets of a universe U and that A and B are disjoint.

5. If $U = \{a,b,c,d,e,f\}$, $A = \{a,b\}$ and $B = \{a,d,f\}$, what is the complement of A? Write the set which is the union of A and B.

6. Show graphically $S = \{x \mid x \text{ is a real number and } x \geq 2\}$.

7. $|x|$ is a symbol for _____.[1] If the universe is the real numbers, find the solution set for $\{x \mid |x - 1| = 3\}$.

8. Given, the universe $U = \{1,2,3,4\}$ and the relation $R = \{(2,3), (2,4), (3,1), (3,2), (3,3)\}$, state its domain, its range.

[1] Complete the statement.

9. Is the relation R of problem 8 a function of U? Explain your answer.

10. Given, the function $F = \{(2;3), (3,4), (4,5), (5,6)\}$, write the inverse of F. Is it a function?

11. Describe in words or make a graph of the set $A \cap B$ where $A = \{(x,y) | x^2 + y^2 > 1\}$, $B = \{(x,y) | x^2 + y^2 < 9\}$, U = real numbers.

12. (a) Given sets A and B, we say that $A \cup B$ is a commutative operation, this means that $A \cup B =$ _____.[1]
(b) State the associative law for intersection with respect to three sets: A, B, and C.

13. Ordered pairs (x,y) and (z,w) are equal if and only if _____.[1]

14. Two sets A and B are equal if and only if _____.[1]

15. Given, the set $S = \{2, 4, 7, 11\}$. Write an example of a set equivalent to S.

16. Let f: $x \longrightarrow y$ be defined by $y = 2x$,
Let g: $x \longrightarrow y$ be defined by $y = x^2 + 1$,
find $f(g(3))$.

[1]Complete the statement.

Achievement Test for Groups and Fields

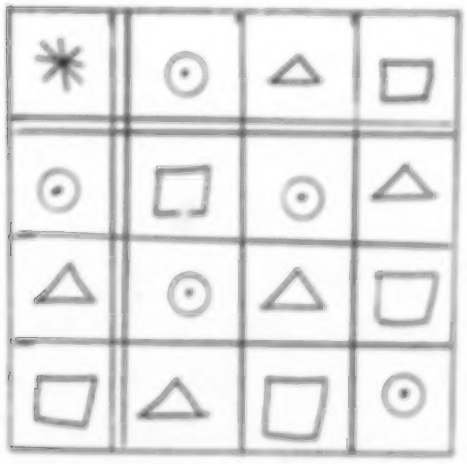
1. Let S and T be sets, where $S = \{a,b,c\}$ and $T = \{1,2\}$
   List all the elements of S x T, (the Cartesian or cross product of S and T).

2. Let $S = \{a,b,c\}$ and consider the following binary relations in S:
   $R_1 = \{(a,b), (b,a), (c,c)\}$
   $R_2 = \{(a,a), (a,b), (b,a), (b,b), (c,c)\}$
   $R_3 = \{(a,b), (b,c), (a,c)\}$
   (a) Which of the above binary relations are reflexive?
   (b) Which of the above binary relations are symmetric?
   (c) Which of the above binary relations are transitive?

3. Which property (or properties) of an equivalence relation is not satisfied by the relation $\leq$ in the set of real numbers?

4. Let $S = \{1,2,3,4\}$ and suppose
   $E = \{(1,1),(2,2),(2,4),(3,3),(4,2),(4,4)\}$
   Note: E is an equivalence relation in S. List the equivalence classes of S with respect to the equivalence relation in S.

5. Let S = {⊙, △, □} be a set having a binary operation * as shown in the table below. (The first factor of a product appears in the first column, the second factor appears in the first row).



Complete the following:

(a) From the table, S has a unique identity element with respect to the binary operation *, namely _____ is the identity element.

(b) Each element of S has a unique inverse with respect to *. The inverse of □ with respect to * is _____.
The inverse of △ with respect to * is _____.
The inverse of ⊙ with respect to * is _____.

(c) From the table, we see that * is a commutative binary operation since for all x, y ∈ S,
_____.

6. Consider the following permutations of degree 3.

$$P = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

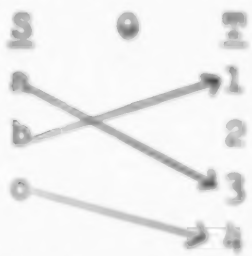
Using the usual definition of permutation multiplication,

(a) Find P · Q

(b) Find Q · P

(c) From your answers to (a) and (b), you see that multiplication of permutations is not _____.

7. Let $S = \{a,b,c\}$ and $T = \{1,2,3,4\}$
Consider the mapping Θ: $S \rightarrow T$ as illustrated below. The illustration indicates the image of each $x \in S$ under the mapping by the use of an arrow.



(1) Θ is a mapping of S into T because _____.
(2) Θ is not a mapping of S onto T because _____.
(3) Θ is a one-to-one mapping of S into T because _____.

8. Let S be the set of integers.
(a) The identity element in S with respect to ordinary addition is _____.
(b) The identity element in S with respect to ordinary multiplication is _____.
(c) The inverse of 7 with respect to addition is _____.
(d) In S, there is no inverse of 7 with respect to multiplication since there is no $x \in S$ such that _____.

9. Let S be a set and let $*$ be a binary operation in S.
Complete the following definitions.
1. $*$ is a commutative binary operation if and only if for all $a,b, \in S$, _____.
2. $*$ is an associative binary operation if and only if for all $a,b,c \in S$, _____.
3. U is called an identity element with respect to $*$ if and only if for every $a \in S$, _____.

4. Suppose S has a unique indentity element with respect to $*$. Call it U.
If $a \in S$, then $a^{-1} \in S$ is an inverse of a with respect to $*$ if and only if _____.

10. The set of integers does not form a group with respect to ordinary multiplication. Why not?

11. Given: S is a group with respect to a binary operation $*$.
Suppose $a,b \in S$. If $a * x = b$ has a solution, show the steps to find the solution.

12. Let S be the set of integers. A binary operation $\oplus$ in S is defined as follows:
For all $a,b \in S$, $a \oplus b = (a + b) + 2$ where $+$ represents ordinary addition. Assume that you have at your disposal all of the properties of addition of integers.
(a) Under the rule of operation, $3 \oplus 5 =$ _____
(b) Prove that $\oplus$ is commutative
(c) Prove that $\oplus$ is associative
(d) Prove that the identity element with respect to $\oplus$ is -2
(e) Prove: if $a \in S$, $(-a-4)$ is the inverse of a with respect to $\oplus$.
(f) The proofs in parts (c), (d) and (e) show that S is a _____ with respect to $\oplus$.

13. Let S be the set of rational numbers having two binary operations defined as follows:
For all a,b $\in$ S, a $\oplus$ b = (a + b) + 2 where + represents ordinary addition.
For all a,b $\in$ S, a $*$ b = (2 $\cdot$ a) $\cdot$ b where $\cdot$ represents ordinary multiplication.

You should note that S is a commutative group with respect to $\oplus$ and S is a commutative group with respect to $*$.

Show that the distributive law of $*$ over $\oplus$ does not hold.

Achievement Test for An Introduction to Probability

1. Six coins are tossed. Find the probability that exactly three heads turn up.

2. $S = \{e_1, e_2, e_3, e_4, e_5\}$ is a sample space for a certain experiment.
   $P(e_1) = .1$, $P(e_2) = .2$, $P(e_3) = .2$,
   $P(e_4) = .4$, $P(e_5) = .1$.
   Consider events A and B where $A = \{e_1, e_3, e_4\}$ and $B = \{e_3, e_5\}$.
   (a) Find $P(A)$
   (b) Find $P(B')$
   (c) Find $P(A \cup B)$
   (d) Find $P(A \cap B)$
   (e) Find $P(A \mid B)$

3. Complete the following definitions:
   (a) Two events, A and B, are mutually exclusive if and only if _____.
   (b) Two events, A and B, are independent if and only if _____.
   (c) Two events, A and B, are dependent if and only if _____.

4. The probability that a certain type of seed will germinate is .8. If 10 such seeds are planted, find the probability that at least 8 will germinate. (Note: you need not do the arithmetic.)

5. An experiment consists of drawing two balls in succession without replacement from a hat containing 3 red balls and 1 white ball. Event A is that the second ball drawn is red.
   (a) Set up a sample space S for this experiment.
   (b) Assign reasonable probabilities to the elements of S.
   (c) Identify A in terms of S.
   (d) Find P (A).

6. Two cards are drawn in succession without replacement from a deck of cards. Find the probability that both are aces.

7. In how many ways can 10 books be arranged on a shelf?

8. If you are allowed to choose three gifts from a set of 10 gifts (no two alike), how many distinct sets of three gifts can you choose?

9. A pair of dice is rolled.
   Let A be the event that a number greater than 9 occurs.
   Let B be the event that an even number occurs.
   Find $P(A)$.
   Find $P(B \mid A)$
   Find $P(A \cap B)$
   Find $P(A \cup B)$

10. Suppose
A is the event that you will get a cold this winter.
B is the event that you will get a sore throat this winter.
In addition, suppose that the following are true:

$P(A) = \frac{7}{10}$

$P(B) = \frac{3}{5}$

$P(B \mid A) = \frac{5}{7}$

Find the probability that you will get a cold or a sore throat this winter.